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LETTER TO THE EDITOR

Combined $\bar{\partial}$ and Riemann–Hilbert inverse methods for integrable non-linear evolution equations in (2+1) dimensions

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Abstract. We give a natural combination of $\bar{\partial}$ and Riemann-Hilbert problem inverse methods for an nth-order scalar spectral problem which solves a number of integrable non-linear evolution equations in two space and one time (2+1) dimensions. The theory embraces the two Kadomtsev-Petviashvili equations and their lump solutions.

It is probably fair to say that the integrable non-linear evolution equations (NEE) in one space and one time (1+1) dimensions are well understood [1, 2]. Consequently, interest has shifted to (2+1) dimensions and a number of physically significant equations have now been solved by inverse spectral methods (ISM). These include the Kadomtsev-Petviashvili (KP) [2, 3], Davey-Stewartson (DS) [4] and three-wave interaction (3WI) [1, 2] equations. Although all of these NEE are completely integrable Hamiltonian systems in the sense of Liouville-Arnold [5], the ISM can become substantially more complicated in (2+1) dimensions: Manakov [6] pioneered the non-local Riemann-Hilbert (RH) problem method which solves KP-I [6, 7], DS-I and 3WI [8], but KP-II [9] and DS-II [8] are solved by the $\bar{\partial}$ method originally due to Beals and Coifman [10]. In the RH method one deals with eigenfunctions analytic (=holomorphic) or meromorphic in separated regions of the complex plane $\mathbb{C}$, but in the $\bar{\partial}$ method the eigenfunctions may be nowhere holo- or meromorphic. Caudrey [11] shows how these two different cases for KP arise as the limit of (local) RH problems for integrable NEE in (1+1) dimensions. In this letter we show how certain (2+1)-dimensional integrable NEE are in general solved by a combination of $\bar{\partial}$ and RH methods. Basically our method is a $\bar{\partial}$ method which exploits patches of local holo- (mero-)morphy. We note that the $\bar{\partial}$ method has recently been extended from 2+1 to some higher-dimensional spectral problems [12] and our results may therefore generalise.

This letter reports the steps of an ISM for an nth-order spectral problem derived from

$$L\Psi(x, y) \equiv \left(\partial_x^n + \sigma\partial_y + \sum_{m=0}^{n-2} u_m\partial_x^m\right)\Psi = 0 \tag{1}$$

where n is an integer $\geqslant 2$; σ is a non-zero complex number and the u_m defined on $\mathbb{R}^2$ are potentials vanishing at infinity. The u_m depend also on time t and a number of (2+1)-dimensional NEE in Lax pair form $[L, A] = 0$, L given by (1), are already reported [13]. When $\sigma = 0$ the problem (1) is the (1+1)-dimensional problem well studied in [14]. Conceptually (1) is a $\bar{\partial}$ problem on the whole of $\mathbb{C}$, in general, but we shall show how it may be thought of as a combination of several $\bar{\partial}$ and RH problems. In fact, patches of local meromorphy of the eigenfunctions can determine certain of the 'lump' solutions of the NEE, or other regions which are not meromorphic but are characterised by a single real root of a certain polynomial $I(\alpha, k)$ (see below) may do so.

Amongst the NEE solved by (1) are the 'complex' KP (CKP) equation

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{2}$$

$\sigma \in \mathbb{C}$, for $n = 2$, and the generalised Sawada-Kotera (gSK) equation [13]

$$(v_t - 5v_x v_{xxx} - \tfrac{5}{3}v_x^2 - v_{xxxxx})_x = 5v_{xxxy} - 5v_{yy} + (v_x v_y)_x \tag{3}$$

for $n = 3$. For simplicity we treat only these two NEE explicitly in this letter. Lax pairs for (2) and (3) have been given already [9, 13]. However, existence of the Lax pair does not itself solve the NEE. In this letter we give the steps of the ISM for (1) for any $n \geq 2$; we define the continuous part of the spectral data as well as the discrete part where this exists, and we then show how these data are inverted. As examples of the application of this ISM to NEE we use it to solve (2) and (3) and finally briefly remark on other NEE solved in this way. The main results reported in this letter are specification of a complete set of spectral data $\mathcal{S}$ for (1) (given in (18) below for any n) and (20) and (21) which together invert that data.

We assume the usual boundary conditions, namely that the u_j vanish 'fast enough' at infinity. Put $\Psi(x, y, k) \equiv \mu(x, y, k) \exp(ikx - i^n\sigma^{-1}k^n y)$ in (1). Then this reduces to the spectral problem on the k plane ($k \in \mathbb{C}$)

$$\left(D^n(k) + \sigma\partial_y - (ik)^n + \sum_{m=0}^{n-2} u_m D^m(k)\right)\mu(x, y, k) = 0 \tag{4}$$

and $D(k) \equiv \partial_x + ik$. We have to find a set of spectral data $\mathcal{S}$ for (4) from which the u_m can be recovered by inversion. To this end the Green function $G(x, y, k)$ satisfying

$$[D^n(k) - (ik)^n + \sigma\partial_y]G(x, y, k) = -\delta(x)\delta(y) \tag{5}$$

is easily found by Fourier transformation as

$$G(x, y, k) = -(2\pi\sigma)^{-1}\,\mathrm{sgn}(y)\int_{-\infty}^{\infty} \exp[i\alpha x + i\Omega(\alpha, k)y]\theta(yI(\alpha, k))\,d\alpha \tag{6}$$

where $\Omega(\alpha, k) \equiv i^{n+1}[(\alpha + k)^n - k^n]\sigma^{-1}$, $I(\alpha, k) = \mathrm{Im}\,\Omega(\alpha, k)$ and $\theta(x) = 1 (x > 0)$, $= 0 (x \leq 0)$; $\mathrm{sgn}(x) = \theta(x) - \theta(-x)$. The zeros of $I(\alpha, k)$ play a crucial role in the theory.

Let the set Γ_e be the zeros of $I(\alpha, k)$ for $k \in \mathbb{C}$ and *every* real α. If $i^{n+1}\sigma^{-1}$ is real this set proves to be the whole of the real line $\mathbb{R}$ of the k plane; otherwise it is the empty set $\varnothing$. $\Gamma_e = \mathbb{R}$ proves to be an essential boundary in the theory: if $\Gamma_e = \mathbb{R}$ we have to define two limiting kernels $G^{\pm}(x, y, k) = \lim_{\zeta \in \mathbb{C}^{\pm} \to k} G(x, y, \zeta)$ for any $k \in \mathbb{C}^{\pm} \cup \mathbb{R}$, where $\mathbb{C}^{\pm}$ are the upper (lower) half planes of $\mathbb{C}$, respectively. The corresponding eigenfunctions $\mu^{\pm}$ of (4) on $\mathbb{C}^{\pm} \cup \mathbb{R}$ are then given by

$$\mu^{\pm}(x, y, k) = 1 + G^{\pm}(x, y, k) * \left[\sum_{m=0}^{n-2} u_m D^m(k)\mu^{\pm}(\xi, \eta, k)\right] \tag{7}$$

where $*$ denotes convolution with respect to both space variables, i.e. $g * h \equiv g(x, y) * h(\xi, \eta) \equiv \iint_{-\infty}^{\infty} g(x - \xi, y - \eta)h(\xi, \eta)\,d\xi\,d\eta$. The eigenfunctions $\mu^{\pm}$ are therefore bounded with respect to the space variables and $\mu^{\pm} \to 1$ as $k \to \infty$. For notational simplicity we shall use μ for μ^+ when $k \in \mathbb{C}^+$ and $\mu = \mu^-$ when $k \in \mathbb{C}^- \cup \mathbb{R}$.

Now let $\alpha_j(k)$ $(j=1,\ldots,N)$ be all of the (continuous in k) complex zeros of the polynomial $I(\alpha,k)\alpha^{-1}$ (the obvious zero $\alpha=0$ of $I(\alpha,k)$ is of no interest). There are N such zeros for every k, while if $\mathrm{i}^{n+1}\sigma^{-1}$ is real $N=n-2$; otherwise $N=n-1$. To simplify here we suppose α_j simple. Then, in general, the k plane divides into regions where there are $0,2,\ldots,N$ real (simple) zeros (N even), or $1,3,\ldots,N$ real zeros (N odd). The $\bar\partial$ method works with the operator $\bar\partial\equiv\partial/\partial\bar k=\frac{1}{2}\,(\partial/\partial k_{\mathrm{R}}+\mathrm{i}\partial/\partial k_{\mathrm{I}})$ with $k=k_{\mathrm{R}}+\mathrm{i}k_{\mathrm{I}}$, and $\bar k=\mathrm{cc}$. The holomorphic functions f satisfy $\bar\partial f=0$; the meromorphic functions also do so (except at the poles). By applying $\bar\partial$ to the Green function G we show that

$$\frac{\partial G}{\partial\bar k}(x,y,k)=\frac{-n(-\mathrm{i})^n}{4\pi|\sigma|^2}\sum_{j=1}^{N}\frac{\tau(\alpha_i)}{|I_\alpha(\alpha_j,k)|}[(\alpha_j+\bar k)^{n-1}-\bar k^{n-1}]\exp[\mathrm{i}\beta_j(x,y,k)] \tag{8}$$

where $I_\alpha\equiv\partial I/\partial\alpha$ ($I_\alpha\neq0$ at $\alpha=\alpha_j$ for simple α_j) and $\beta_j(x,y,k)=\alpha_j(k)x+\Omega(\alpha_j(k),k)y$; $\tau(z)$, $z\in\mathbb{C}$, is a function which is unity $z\in\mathbb{R}$ and is zero otherwise. From (8) and (7) together we then find the 'defect of holomorphy'

$$\frac{\partial\mu}{\partial\bar k}(x,y,k)=\sum_{j=1}^{N}T_j(k)\exp[\mathrm{i}\beta_j(x,y,k)]\mu(x,y,k+\alpha_j(k)) \tag{9}$$

for $k\in\mathbb{C}/\Gamma_e$; the T_j are defined by

$$T_j(k)=n[(-1)^n/4\pi|\sigma|^2]\tau(\alpha_j)\{[(\alpha_j+\bar k)^{n-1}-\bar k^{n-1}]/|I_\alpha(\alpha_j,k)|\}$$
$$\times\int\!\!\int_{-\infty}^{\infty}\exp[\mathrm{i}\beta_j(\xi,\eta,k)]\sum_{m=0}^{n-2}u_m(\xi,\eta)D^m(k)\mu(\xi,\eta,k)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{10}$$

for $1\leqslant j\leqslant N$ and constitute a part of the continuous part of the spectral data $\mathscr{S}$ we are looking for. Evidently, in regions where there are no real zeros α_j, the T_j vanish and $\partial\mu/\partial\bar k=0$ there.

We are obliged to omit in this letter a number of intermediate steps including the demonstration of the 'symmetry' [9] of the kernel G, $G(x,y,k)\exp[\mathrm{i}\beta_j(x,y,k)]=G(x,y,k+\alpha_j)$ used in deriving (9): the final results can all be verified directly. These results include identification of other continuous spectral data $f(l,k)$, discrete spectral data γ_j, their time evolution and their inversion. We introduce the $f(l,k)$ next.

If $\Gamma_e=\varnothing$ the $f(l,k)$ vanish. If $\Gamma_e=\mathbb{R}$ we measure the jump over Γ_e by

$$\Delta\mu(x,y,k)=\mu^+(x,y,k)-\mu^-(x,y,k)=\int\!\!\int_{-\infty}^{\infty}f(l,k)\mu^-(x,y,l)e(l,k,x,y)\,\mathrm{d}l \tag{11}$$

where $e(l,k,x,y)\equiv\exp[\mathrm{i}(l-k)x-\mathrm{i}^n\sigma^{-1}(l^n-k^n)y]$. Let $\varepsilon_n=\mathrm{sgn}(\mathrm{i}^{n+1}\sigma^{-1})$ (which makes sense since $\Gamma_e\neq\varnothing$). We introduce functions $W(l,k,x,y)$ and $T(k,l)$ by

$$W(l,k,x,y)=e(l,k,x,y)+G^-(x,y,k)*\left[\sum_{m=0}^{n-2}u_mD^m(k)\,W(l,k,\xi,\eta)\right] \tag{12}$$

and

$$T(k,l)=\varepsilon_n(2\pi\sigma)^{-1}\,\mathrm{sgn}(k^{n-1}-l^{n-1})\int\!\!\int_{-\infty}^{\infty}e(k,l,\xi,\eta)\sum_{m=0}^{n-2}u_mD^m(k)\mu^+(\xi,\eta,k)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{13}$$

The superposition principle shows that $\Delta\mu(x, y, k) = \int_{-\infty}^{\infty} T(k, l)\, W(l, k, x, y)\, \mathrm{d}l$. However, these definitions also enable us to represent $f(l, k) = f_+(l, k) + f_-(-l, k)$ with

$$f_\pm(l, k) = \varepsilon_n^\pm (2\pi\sigma)^{-1}\, \mathrm{sgn}(l^{n-1} - kl^{n-2}) \iiint_{-\infty}^{\infty} T(k, s)$$

$$\times \sum_{m=0}^{n-2} u_m(\xi, \eta) D^m(\pm l) e(l, \pm l, \xi, \eta)\, W(s, l, \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}s \tag{14}$$

in which $\varepsilon_n^- = 0$ (n even), otherwise $\varepsilon_n^\pm = \varepsilon_n$. Moreover, if we define further 'inverse' data $T_\pm$ by

$$T_\pm(l, k) = (2\pi\sigma)^{-1} \sum_{m=0}^{n-2} \theta(\pm\varepsilon_n\, \mathrm{sgn}(k^{n-1} - l^{n-1}))$$

$$\iint_{-\infty}^{\infty} u_m(\xi, \eta) e(k, l, \xi, \eta) D^m(k) \mu^\pm(\xi, \eta, k)\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{15}$$

the spectral data $f(l, k)$ can be simply solved for this inverse data through an integral equation

$$f(l, k) + \int_{-\infty}^{\infty} T_-(l, s) f(s, k)\, \mathrm{d}s = T_+(l, k) - T_-(l, k) \tag{16}$$

($(l, k) \in \mathbb{R}^2$). This way the $f(l, k)$ are calculated from the u_m.

We now turn to the discrete data: in the different regions $\mathscr{D}$ μ may be meromorphic with poles (here assumed simple for simplicity) when and only when there are no real zeros α_j in $\mathscr{D}$. For $k_l \in \mathscr{D}$ and $\mathscr{D}$ meromorphic we find we can define discrete data γ_l through

$$\lim_{k \in \mathscr{D} \to k_l} [\mu(x, y, k) - \psi_l(x, y)(k - k_l)^{-1}] = -\mathrm{i}(x + n\sigma^{-1}\mathrm{i}^{n+1} k_l^{n-1} y + \gamma_l)\psi_l(x, y) \tag{17}$$

where $\psi_l(x, y)$ is the residue at the pole k_l. This definition then extends to those $\mathscr{D}$ where there are real zeros. However, we then find for consistency that there are only two cases: either $\sigma \in \mathbb{R}$ (and N is odd) and there is one real zero $\alpha_1 = -(k + \bar{k})$, or there are no real zeros (and N is even).

To summarise: we have now defined a set of spectral data for the spectral problem (4)

$$\mathscr{S} = \{T_j(z), z \in \mathbb{C}/\Gamma_e, j = 1, \ldots, N; f(l, k) \in \mathbb{R}; k_l, \gamma_l, l = 1, \ldots, N'\} \tag{18}$$

which can be calculated from the potentials u_m, $0 \leqslant m \leqslant n-2$. At most there is only the one $T_j(z)$, $T_1(z)$, non-vanishing in regions $\mathscr{D}$ containing any of the k_l. The real line $\mathbb{R}$ is an essential boundary Γ_e and if $\Gamma_e = \varnothing$ the $f(l, k)$ vanish. We need take no account of the boundaries of the regions where some of the T_j jump, since the T_j are bounded in their neighbourhoods and appear later only under integral signs. We have still to show that the set $\mathscr{S}$ is sufficient to determine all the u_m and to invert $\mathscr{S}$ to regain the u_m.

Since $\mu(x, y, k)$ is continuous in these neighbourhoods in $\mathbb{C}/\Gamma_e$ we can use the generalised Cauchy formula (derived from Stokes's theorem) [9]

$$g(k) = \frac{1}{2\pi\mathrm{i}} \iint_\omega \frac{\partial g/\partial\bar{z}}{z - k}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{1}{2\pi\mathrm{i}} \int_{\partial\omega} \frac{g(z)\, \mathrm{d}z}{z - k} \tag{19}$$

for an arbitrary simply connected region $\omega \subset \mathbb{C}$ to obtain the following linear integral equation:

$$\mu(x,y,k)=1+\frac{1}{2\pi \mathrm{i}}\iint\limits_{\mathbb{C}}\sum_{j=1}^{N}\frac{T_j(z)\exp[\mathrm{i}\beta_j(x,y,z)]}{z-k}\,\mu(x,y,z+\alpha_j(z))\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$$
$$+\frac{1}{2\pi\mathrm{i}}\iint\limits_{-\infty}^{\infty}\frac{f(l,s)e(l,s,x,y)\mu(x,y,l)\,\mathrm{d}s\,\mathrm{d}l}{s-k+\mathrm{i}0}+\sum_{l=1}^{N'}\frac{\psi_l}{k-k_l} \tag{20}$$

where $\mathrm{d}z\wedge\mathrm{d}\bar{z}=-2\mathrm{i}\mathrm{d}z_{\mathrm{R}}\mathrm{d}z_{\mathrm{I}}$, and we invoke $\mu(k)\to 1$ as $k\to\infty$. Furthermore, under the limit $k\to k_j$, (20) becomes with the help of (17) $(1\leqslant j\leqslant N')$

$$-\mathrm{i}(x+\sigma^{-1}n\mathrm{i}^{n+1}k_j^{n-1}y+\gamma_j)\psi_j$$
$$=1+\frac{1}{2\pi\mathrm{i}}\iint\limits_{\mathbb{C}}\sum_{m=1}^{N}T_m(z)\exp[\mathrm{i}\beta_m(x,y,z)]$$
$$\times\mu(x,y,z+\alpha_m(z))(z-k_j)^{-1}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$$
$$+\frac{1}{2\pi\mathrm{i}}\iint\limits_{-\infty}^{\infty}\frac{f(l,s)e(l,s,x,y)}{s-k_j+\mathrm{i}0}\,\mu(x,y,l)\,\mathrm{d}s\,\mathrm{d}l+\sum_{l=1,l\neq j}^{N'}\frac{\psi_l}{k_j-k_l}. \tag{21}$$

The two equations (20) and (21) provide the fundamental integral equations for the inverse spectral transform from $\mathcal{S}$ to μ: the function μ then determines the u_m by using the expansion $\mu\sim 1+\mu_1(x,y)k^{-1}+\ldots$ derived from (7) put into (4): $\mu_1=\lim_{k\to\infty}(\mu-1)k$.

We turn to the application of these equations to the solution of integrable NEE: we take equations (2) and (3) as examples. We have to find the zeros of $I(\alpha,k)\alpha^{-1}$. When $n=2$ they are given by $\alpha=-2(k+\bar{k}\mathrm{e}^{\mathrm{i}\Phi})/(1+\mathrm{e}^{\mathrm{i}\Phi})$ $(\Phi=2\arg\sigma+n\pi)$ so there are no real zeros iff $\mathrm{e}^{\mathrm{i}\Phi}=-1$. There are then no T_j anywhere in $\mathbb{C}$, the integral over $\mathbb{C}$ vanishes in (20) and (21), and the inverse spectral problem reduces to the solution of the usual non-local RH problem which solves the KP-I equation (this is equation (2) with $\sigma=\mathrm{i}$ [3, 6, 7, 9]). Because integrals over $\mathbb{C}$ do not appear, (20) and (21) can be solved on the smaller region $\mathbb{R}$ of $\mathbb{C}$. The contributions of simple poles in meromorphic regions $\mathbb{C}^+$, $\mathbb{C}^-$ yield the known lump solutions [7].

Otherwise $N=1$, and there is always one real root for any other Φ. This includes the KP-II equation with $\sigma=-1$ [9] and $\alpha_1=-(k+\bar{k})$. Since $T_1(k)\neq 0$, μ is nowhere meromorphic: $\Gamma_e=\varnothing$, the $f(l,k)$ vanish and (20) and (21) reduce to the usual $\bar{\partial}$ solution [9]. Lump solutions are still possible in principle for $\sigma=-1$ and it is therefore noteworthy that if, with $f(l,k)=0$, we also formally set $T_1(k)=0$, we recover the singular one-lump solution of [15] by taking $N=2$ with $k_2=\bar{k}_1$ and solving (21) algebraically.

The CKP equation (2) has the Lax representation $[L,A]=0$ with

$$L\equiv\partial_x^2+\sigma\partial_y+u\qquad A\equiv 4\partial_x^3+6u\partial_x+3\left(u_x-\sigma\int_{-\infty}^{x}u_y\,\mathrm{d}x\right)+\partial_t \tag{22}$$

and from this the evolution of the spectral data is found to be (cases $\sigma = \mathrm{i}$, $\sigma = -1$ only)

$$\frac{\partial T}{\partial t}(z,t) = -4\mathrm{i}(z^3+\bar{z}^3)T_1(z,t) \qquad \frac{\partial f}{\partial t}(l,k,t) = 4\mathrm{i}(l^3-k^3)f(l,k,t)$$
$$\frac{\partial k_j}{\partial t} = 0 \qquad \frac{\partial \gamma_j}{\partial t} = 12\,k_j^2. \tag{23}$$

T_1 vanishes for $\sigma = i$ (KP-I) and both the $f(l,k)$ and (for non-singular lumps) the k_j and γ_j vanish for $\sigma = -1$ (KP-II). We have proved elsewhere [5] that equations (23) are just Hamilton's equations for canonical combinations of these spectral data: $|T_1(z,t)|^2$ are action variables for KP-II and $|f(l,k,t)|^2$ with the k_j are action variables for KP-I; it is plain from (23) that they are all constants of the motion. Finally, by (20) and (21) μ is found from these data, $\mathcal{S}$, and u is then found from $u = -2\mathrm{i}\partial_x \lim_{k\to\infty} [\mu(x,y,t)-1]k$.

The case $n=3$ is analysed similarly: the gSK equation (3) has Lax representation $[L,A]=0$ with $L \equiv \partial_x^3 + u\partial_x + \sigma\partial_y$, $u = v_x$, and $\sigma = 1$. Thus $\Gamma_e = \mathbb{R}$, $N=1$ and there is one real zero (which is $\alpha_1 = -(k+\bar{k})$). The spectral data $\mathcal{S} = \{T_1(z,t); f(l,k,t); k_j, \gamma_j, 1 \leqslant j \leqslant N'\}$ and there are lump solutions. The data $\mathcal{S}$ prove to evolve as

$$\frac{\partial T_1}{\partial t}(z,t) = 9\mathrm{i}(z^5+\bar{z}^5)T_1(z,t) \qquad \frac{\partial f}{\partial t}(l,k,t) = 9\mathrm{i}(k^5-l^5)f(l,k,t)$$
$$\frac{\partial k_j}{\partial t} = 0 \qquad \frac{\partial \gamma_j}{\partial t} = -45k_j^4 \tag{24}$$

with all of $|T_1(z,t)|$, $|f(k,l,t)|$ and the k_j constants of the motion. This example is therefore a genuine combination of RH and $\bar{\partial}$ problems in the sense of this letter: μ is found similarly as for KP so is given by

$$u(x,y,t) = \partial_x\Bigg(\frac{3}{2\pi}\iint_{\mathbb{C}} T_1(z,t)\exp(-\mathrm{i}(z+\bar{z})x - \mathrm{i}(z^3-\bar{z}^3)y]\mu(x,y,t,-\bar{z})\,\mathrm{d}\bar{z}\wedge\mathrm{d}z$$
$$+\frac{3}{2\pi}\iint_{-\infty}^{\infty} f(l,s)\exp[\mathrm{i}(l-s)x + \mathrm{i}(l^3-s^3)y]\mu(x,y,t,l)\,\mathrm{d}s\,\mathrm{d}l$$
$$-3\mathrm{i}\sum_{l=1}^{N'}\psi_l(x,y,t)\Bigg). \tag{25}$$

The corresponding equation with $\sigma = i$ has $\Gamma_e = \varnothing$, $N=2$ and regions with two and zero real roots. It can have poles only in the latter and μ is meromorphic there: $\mathcal{S} = \{T_1, T_2; k_j, \gamma_j, 1 \leqslant j \leqslant N'\}$; $f=0$.

Finally we note that the generalised Gardner equation $u_{xxx} + 6\beta u u_x - \frac{3}{2}\alpha^2 u_x^2 + 3\partial_x^{-1}u_{yy} - 3\alpha u_x \partial_x^{-1} u_y + u_t = 0$ has a Lax pair with, for $n=2$, the term $u_{n-1}\partial_x^{n-1}$ added to L in (1) [13]. We shall report other methods to handle this case. For $\beta = 0$ it contains a modified KP equation of considerable current interest [16].

A comprehensive account of the ISM for (1) reported in this letter and of its applications to integrable NEE will be given elsewhere.

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